JOURNAL OF APPROXIMATION THEORY 7, 293–301 (1973)

Interpolating Subspaces in l_1-Spaces

J. H. BIGGS, F. R. DEUTSCH,* R. E. HUFF, P. D. MORRIS, J. E. OLSON

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by E. W. Cheney

Received December 7, 1970

1. INTRODUCTION

The notion of an *interpolating subspace* of a normed linear space was introduced in [1] as a generalization of a Haar subspace in $C[a, b]$. A very lengthy and nonconstructive proof was given in [1] to show that the *real* spaces l_1 and l_1^m contain interpolating subspaces of every dimension. In this paper in Section 2, we give a *constructive* proof which is substantially shorter. In Section 3, we show that quite the opposite is true for the *complex* spaces l_1 and l_1^m. Indeed (Theorem 3.6): no proper subspace M of dimension greater than one is interpolating for any point outside M. (It is clear that the unit vector $(1, 0,...)$ in l_1 or l_1^m spans a one-dimensional interpolating subspace.)

Our terminology conforms to that of [1]. Let M be an n-dimensional subspace of a normed linear space X. If x is in X, we say that M is *interpolating for* x if, for each set of n functionals $x_1^*,..., x_n^*$ in ext $S(X^*)$, there is a $y \in M$ such that $x_i^*(y) = x_i^*(x)$ $(i = 1,..., n)$. (Here ext $S(X^*)$ denotes the set of extreme points of the unit ball of X^*.) M is an *interpolating subspace* of X if and only if M is interpolating for every $x \in X$. (Although it will not be needed in the sequel, the following fact is of independent interest: if M is interpolating for some $x \in X$, then M is an interpolating subspace of the linear span of M and x; hence by [1, Theorem 2.2], x has a unique best approximation in M.)

Recall that $l_1^* = l_\infty$, $(l_1^m)^* = l_\infty^m$, and that $x^* = (\sigma_1, \sigma_2, ...) \in l_\infty$ (respectively, $x^* = (\sigma_1, \sigma_2, ..., \sigma_m) \in l_\infty^m$) is in ext $S(l_1^*)$ (respectively, ext $S[(l_1^m)^*]$) if and only if $|\sigma_i| = 1$ for all i.

* This author was supported by a grant from the National Science Foundation.

2. A Constructive Proof of the Existence of Interpolating Subspaces in the Real Spaces l_1 and l_1^m

Consider first the space l_1. Fix an arbitrary $n \geqslant 1$. Set $x_i = (x_{i1}, x_{i2}, \dots)$ where

$$x_{ij} = r^{2^{nj+i}} \qquad (i = 1,\dots, n; \quad j = 1, 2,\dots)$$

and

$$0 < r < (1 + n^{n/2})^{-1}$$

We shall show that $x_1, \dots, x_n$ *is a basis for an n-dimensional interpolating subspace in* l_1. This is equivalent to

$$\det [x_i^*(x_j)] \neq 0 \text{ for every set of } n \text{ linearly independent functionals } x_i^* \text{ in } \operatorname{ext} S(l_1^*). \tag{I}$$

Let $x_i^* = (\sigma_{i1}, \sigma_{i2}, \dots)$ $(i = 1,\dots, n)$ be linearly independent functionals in $\operatorname{ext} S(l_1^*)$. Now

$$\det[x_i^*(x_j)] = \begin{vmatrix} \sum \sigma_{1j}x_{1j} & \sum \sigma_{1j}x_{2j} & \cdots & \sum \sigma_{1j}x_{nj} \\ \cdots & & & \\ \sum \sigma_{nj}x_{1j} & \sum \sigma_{nj}x_{2j} & \cdots & \sum \sigma_{nj}x_{nj} \end{vmatrix}$$

$$= \sum_{j_1,\dots,j_n=1}^{\infty} x_{1j_1}x_{2j_2} \cdots x_{nj_n} D(j_1, \dots, j_n),$$

where

$$D(j_1, \dots, j_n) = \begin{vmatrix} \sigma_{1j_1} & \sigma_{1j_2} & \cdots & \sigma_{1j_n} \\ \cdots & & & \\ \sigma_{nj_1} & \sigma_{nj_2} & \cdots & \sigma_{nj_n} \end{vmatrix} \tag{1}$$

Substituting for x_{ij}, we get

$$\det[x_i^*(x_j)] = \sum_{j_1,\dots,j_n=1}^{\infty} r^{2^{nj_1+1}+2^{nj_2+2}+\dots+2^{nj_n+n}} D(j_1, \dots, j_n). \tag{2}$$

Suppose $\{j_1, \dots, j_n\}$ and $\{j_1', \dots, j_n'\}$ are any two sets of positive integers such that

$$2^{nj_1+1} + 2^{nj_2+2} + \cdots + 2^{nj_n+n} = 2^{nj_1'+1} + 2^{nj_2'+2} + \cdots + 2^{nj_n'+n}. \tag{3}$$

Because of the special form of the exponents in expression (3), and because every integer has a unique binary expansion, it follows that $j_i = j_i'$ for

$i = 1,\dots, n$. In particular then, distinct ordered arrays $\{j_1 ,\dots, j_n\}$ give rise to distinct powers of r in (2). Hence *each nonzero coefficient of the right side of expression* (2), *regarded as a power series in* r, *is a determinant* $D(j_1 ,\dots, j_n)$.

LEMMA 2.1. *The coefficients $D(j_1 ,\dots, j_n)$ are all integers and at least one is nonzero. Moreover,* $| D(j_1 ,\dots, j_n)| \leqslant n^{n/2}$.

Proof. Since $\sigma_{ij} = \pm 1$ for every i and j, it follows from (1) that $D(j_1 ,\dots, j_n)$ is an integer. Further, by Hadamard's determinant inequality, $| D(j_1 ,\dots, j_n)| \leqslant n^{n/2}$. Since the vectors $x_i^* = (\sigma_{i1} , \sigma_{i2} ,\dots)$ $(i = 1,\dots, n)$ are linearly independent, the rank of the n by ∞ matrix having these vectors as rows is n. Hence $D(j_1 ,\dots, j_n) \neq 0$ for some $j_1 ,\dots, j_n$.

LEMMA 2.2. *Let $f(r) = \sum_0^\infty a_n r^n$ be a power series whose coefficients are integral, not all zero, and $| a_n | \leqslant M$. If $0 < \xi < (1 + M)^{-1}$, then $f(\xi) \neq 0$.*

Proof. Let N denote the smallest integer n such that $a_n \neq 0$. Then

$$\begin{aligned} |f(\xi)| &= | a_N\xi^N + a_{N+1}\xi^{N+1} + \cdots | \\ &\geqslant | a_N | \, | \xi |^N - \sum_{N+1}^{\infty} | a_n | \, | \xi |^n \\ &\geqslant | \xi |^N - M \frac{| \xi |^{N+1}}{1 - | \xi |} = \frac{| \xi |^N}{1 - | \xi |} [1 - (1 + M) | \xi |] \\ &> 0. \end{aligned}$$

Now for each set of n linearly independent functionals $x_i^* \in \operatorname{ext} S(l_1^*)$, the expression (2) is a power series in r with coefficients $D(j_1 ,\dots, j_n)$ which satisfy the hypothesis of Lemma 2.2 with $M = n^{n/2}$. Since $0 < r < (1 + n^{n/2})^{-1}$, it follows from Lemma 2.2 that (I) holds, and so we have that $x_1 ,\dots, x_n$ *spans an n-dimensional interpolating subspace in* l_1.

For the case l_1^m, fix an arbitrary integer $1 \leqslant n \leqslant m$. Set

$$x_i = (x_{i1} , x_{i2} ,\dots, x_{im}) \qquad (i = 1,\dots, n),$$

where (as before)

$$x_{ij} = r^{2^{nj+i}}$$

and $0 < r < (1 + n^{n/2})^{-1}$. Then exactly the same proof as above shows that $x_1 ,\dots, x_n$ *spans an n-dimensional interpolating subspace in* l_1^m.

3. The Nonexistence of Interpolating Subspaces of Dimension Greater Than One in the Complex Spaces l_1 and l_1^m.

We first prove four lemmas concerning matrices. For these lemmas, unless otherwise stated, lower case letters will denote complex numbers.

Call two numbers a and b *parallel* if there exist real numbers x and y, not both zero, such that $xa + yb = 0$.

LEMMA 3.1. *If $B = (b_{ij})$ is a 3×2 matrix whose first two rows are linearly independent, then there is a nonzero vector (c_1, c_2, c_3) in the column space of B and a solution z_1, z_2, z_3, to the equation $z_1c_1 + z_2c_2 + z_3c_3 = 0$ such that $|z_i| = 1$ and either*

(i) *some two of the numbers z_ic_i are not parallel, or*

(ii) $c_3 = 0$.

Proof. First perform column operations to bring B into the form

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{bmatrix}.$$

If $|a| = |b|$, then the column space contains a vector $(c_1, c_2, 0)$ with $|c_1| = |c_2| = 1$, in which case we take $z_1 = -c_2/c_1$, $z_2 = z_3 = 1$.

If $a = 0, b \neq 0$, the column space contains the vector $(c_1, c_2, c_3) = (d, 1, b)$ for arbitrary d. If b is real, take $d = -(i + b)$ and $(z_1, z_2, z_3) = (1, i, 1)$. Thus $z_1c_1 + z_2c_2 + z_3c_3 = 0$ and $z_2c_2 = i$ and $z_3c_3 = b$ are not parallel. If b is not real, take $z_1 = z_2 = z_3 = 1$ and $d = -(1 + b)$.

The case $a \neq 0$, $b = 0$ is similar to the case above. Assume therefore that $a \neq 0$, $b \neq 0$, and $|a| \neq |b|$. In particular, either $b \neq -1$ or $a \neq -1$. By symmetry, it suffices to assume $b \neq -1$. Choose $z \neq -a$, $|z| = 1$, such that $(1 + b)z/(a + z)$ is not real. The column space contains the vector $(c_1, c_2, c_3) = (d, 1, da + b)$ for arbitrary d. Take $d = -(1 + b)/(a + z)$ and $(z_1, z_2, z_3) = (z, 1, 1)$. Thus $z_1c_1 + z_2c_2 + z_3c_3 = 0$, and z_1c_1 is not parallel to z_2c_2 since $z_1c_1 = zd$ is not real but $z_2c_2 = 1$.

LEMMA 3.2. *If $B = (b_{ij})$ is an $m \times 2$ matrix ($m \geqslant 3$) of rank 2, then there is a nonzero vector $(c_1, c_2, \ldots, c_m)$ in the column space of B and a solution $z_1, z_2, \ldots, z_m$ to the equation $z_1c_1 + z_2c_2 + \cdots + z_mc_m = 0$ such that $|z_i| = 1$ for each i and either*

(i) *some two of the z_ic_i are not parallel, or*

(ii) *all but two of the c_i are zero.*

Proof. Rearrange the notation so that the first two rows of B are linearly independent. Let

$$B_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ a & b \end{bmatrix},$$

where $a = b_{31} + \cdots + b_{m1}$, $b = b_{32} + \cdots + b_{m2}$, and apply Lemma 3.1: there is a nonzero vector (c_1, c_2, c) in the column space of B_1 and a solution (z_1, z_2, z) to the equation $z_1c_1 + z_2c_2 + zc = 0$ such that $|z_1| = |z_2| = |z| = 1$ and either $c = 0$ or some two of z_1c_1, z_2c_2, zc are not parallel. Let $(c_1, c_2, \ldots, c_m)$ be the corresponding vector in the column space of B (where $c = c_3 + \cdots + c_m$).

If some two of z_1c_1, z_2c_2, zc are not parallel, take $z_3 = \cdots = z_m = z$. It follows that some two of $z_1c_1, z_2c_2, \ldots, z_mc_m$ are not parallel.

Now suppose $c = 0$. Then $c_1 \neq 0$. If $c_3 = \cdots = c_m = 0$, take $z_3 = \cdots = z_m = z$. If $c_j \neq 0$ for some $3 \leqslant j \leqslant m$, choose $|z'| = 1$ such that z_1c_1 and $z'c_j$ are not parallel, and take $z_3 = \cdots = z_m = z'$.

LEMMA 3.3. *If $d_1 + d_2 + \cdots + d_m = 0$ $(m \geqslant 3)$ and some two of the d_i are not parallel, then there is an $(m-1) \times m$ matrix $W = (w_{ij})$ of rank $m-1$, with $|w_{ij}| = 1$, such that*

$$W \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1}$$

Proof. Proceed by induction on m. For $m = 3$, arrange the notation so that d_1 is not parallel to d_2. We can assume without loss of generality that $d_1 + d_2 = 1$. Hence neither d_1 nor d_2 is real. Let w, z be the solution of the equation $wd_1 + zd_2 = 1$ which satisfies $|w| = |z| = 1$ and $w \neq z$ (i.e., take $w = \bar{d}_1/d_1$, $z = \bar{d}_2/d_2$). The matrix

$$W = \begin{bmatrix} 1 & 1 & 1 \\ w & z & 1 \end{bmatrix}$$

is then a solution to Eq. (1).

Assume now that $m \geqslant 4$ and that Lemma 3.3 is true for $m - 1$. If some d_i is zero, we can assume that $d_m = 0$. By the induction hypothesis, there is an $(m-2) \times (m-1)$ matrix $W_1 = (w_{ij})$ of rank $m-2$ such that

$$W_1 \begin{bmatrix} d_1 \\ \vdots \\ d_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence the matrix

$$W = \begin{bmatrix} w_{11} & \cdots & w_{1,m-1} & 1 \\ \vdots & & & \vdots \\ w_{m-2,1} & \cdots & w_{m-2,m-1} & 1 \\ w_{11} & \cdots & w_{1,m-1} & -1 \end{bmatrix}$$

has rank $m - 1$ and satisfies Eq. (1).

Assume therefore that all d_i are nonzero. Since $m \geqslant 4$, the d_i are all nonzero, and some two of the d_i are not parallel, it follows that there are distinct indices i, j, k such that d_i is not parallel to d_j and $d_i + d_j$ is not parallel to d_k. Without loss of generality, we may assume that d_1 is not parallel to d_2 and $d_1 + d_2$ is not parallel to some d_k ($k \geqslant 3$). Also, we may assume that $d_1 + d_2 = 1$, and hence that d_1 and d_2 are not real. As in the case $m = 3$, let w, z be the solution to the equation $wd_1 + zd_2 = 1$ which satisfies $|w| = |z| = 1$ and $w \neq z$. By the induction hypothesis, there is an $(m-2) \times (m-1)$ matrix $W_1 = (w_{ij})$ such that $|w_{ij}| = 1$, W_1 has rank $m - 2$, and

$$W_1 \begin{bmatrix} d_1 + d_2 \\ d_3 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix

$$W = \begin{bmatrix} w_{11} & w_{11} & w_{12} & \cdots & w_{1,m-1} \\ \vdots & & & & \\ w_{m-2,1} & w_{m-2,1} & w_{m-2,2} & \cdots & w_{m-2,m-1} \\ ww_{11} & zw_{11} & w_{12} & \cdots & w_{1,m-1} \end{bmatrix}$$

has rank $m - 1$ since W_1 has rank $m - 2$ and the last row of W is not a linear combination of the first $m - 2$ rows. Clearly, W is a solution to Eq. (1). This completes the induction step and proves the Lemma.

LEMMA 3.4. *If $B = (b_{ij})$ is an $m \times 2$ matrix of rank 2 ($m \geqslant 3$), then there is a nonzero vector $(c_1, c_2, \ldots, c_m)$ in the column space of B and an $(m - 1) \times m$ matrix $E = (\sigma_{ij})$ of rank $m - 1$ such that $|\sigma_{ij}| = 1$ and*

$$E \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2}$$

Proof. By Lemma 3.2 there is a nonzero vector $(c_1, c_2, \ldots, c_m)$ in the column space of B and a solution $z_1, z_2, \ldots, z_m$ to the equation

$z_1c_1 + z_2c_2 + \cdots + z_mc_m = 0$ such that $|z_i| = 1$ for each i and which satisfies condition (i) or (ii) of that Lemma.

If (i) holds, some two of the z_ic_i are not parallel. Hence by Lemma 3.3 there is an $(m-1) \times m$ matrix $W = (w_{ij})$ of rank $m-1$ such that $|w_{ij}| = 1$ for each i and j and

$$W \begin{bmatrix} z_1c_1 \\ z_2c_2 \\ \vdots \\ z_mc_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The $(m-1) \times m$ matrix $E = (w_{ij}z_j)$ has rank $m-1$ and is a solution to Eq. (2).

If (ii) holds, we may assume without loss of generality that $c_3 = c_4 = \cdots = c_m = 0$. Let $E_1 = (\sigma_{ij})$ be any $(m-1) \times (m-1)$ matrix of rank $m-1$ such that $|\sigma_{ij}| = 1$ and $\sigma_{11} = \sigma_{21} = \cdots = \sigma_{m-1,1} = z_2$. Then the $(m-1) \times m$ matrix

$$E = \left[\begin{array}{c|c} z_1 & \\ z_1 & \\ \vdots & E_1 \\ z_1 & \end{array}\right]$$

has rank $m-1$ and is a solution to Eq. (2).

We shall now use Lemma 3.4 to establish the nonexistence of interpolating subspaces in the complex spaces l_1 and l_1^m.

If M is a finite-dimensional subspace of a normed linear space X and if $A \subset X^*$, then we write $\dim_M A$ for the dimension of the subspace of M^* spanned by the restrictions of the members of A to M.

LEMMA 3.5. *Let M be an n-dimensional subspace of the complex space l_1^m $(1 < n < m)$. Then there exists a set $A = \{x_1^*, x_2^*, \ldots, x_{m-1}^*\}$ of $m-1$ linearly independent functionals in* ext $S((l_1^m)^*)$ *such that* $\dim_M A < n$.

Proof. By Lemma 3.4, there exist $m-1$ linearly independent functionals $x_1^*, \ldots, x_{m-1}^*$ in ext $S((l_1^m)^*)$ and a nonzero element $y \in M$ such that $x_i^*(y) = 0$ for $i = 1, \ldots, m-1$. Letting $A = \{x_1^*, \ldots, x_{m-1}^*\}$, it follows that $\dim_M A < n$.

THEOREM 3.6. *Let M be an n-dimensional subspace $(n > 1)$ of complex l_1 (or l_1^m). Then M is not interpolating for any point outside of M. In particular, the complex spaces l_1 and l_1^m contain no proper interpolating subspace of dimension greater than one.*

Proof. Assume first that $M \subset l_1^m$ and suppose, on the contrary, that M is interpolating for some $x_0 \in l_1^m \sim M$. Let M_0 be the linear span of M and x. By Lemma 3.5, there is a set A of $m - 1$ linearly independent functionals in ext $S((l_1^m)^*)$ such that $\dim_M A < n$. Since M is interpolating for x_0, it follows that $\dim_{M_0} A = \dim_M A < n$. Let F be a subspace of l_1^m complementary to M_0, so that $\dim F = m - n - 1$. Then $\dim_F A \leqslant m - n - 1$ so that

$$\dim(\text{span } A) \leqslant \dim_{M_0} A + \dim_F A < n + m - n - 1 = m - 1,$$

which is absurd.

Assume next that $M \subset l_1$ and suppose, on the contrary, that M is interpolating for some $y = (y_1, y_2, ...)$ in $l_1 \sim M$. We choose an integer k so that the map $(x_1, x_2, ...) \to (x_1, x_2, ..., x_k)$ is one-to-one on the linear span M_0 of M and y. Define $L: l_1 \to l_1^{k+1}$ by

$$L(x_1, x_2, ...) = \left(x_1, x_2, ..., x_k, \sum_{k+1}^{\infty} x_i\right).$$

Then L is one-to-one on M_0. Let $M' = L(M)$ and $y' = L(y)$. Let $\tilde{x}_1^*, ..., \tilde{x}_n^*$ be functionals in ext $S((l_1^{k+1})^*)$ with

$$\tilde{x}_i^* = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{i,k+1}) \qquad (i = 1, ..., n).$$

Define

$$x_i^* = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{ik}, \sigma_{i,k+1}, \sigma_{i,k+1}, ...)$$

$(i = 1, ..., n)$. Then x_i^* is in ext $S(l_1^*)$. Therefore there exists $w = (w_1, w_2, ...) \in M$ such that $x_i^*(w) = x_i^*(y)$ for $i = 1, ..., n$. Letting $w' = L(w)$, we see that

$$\tilde{x}_i^*(w') = x_i^*(w) = x_i^*(y) = \tilde{x}_i^*(y') \qquad (i = 1, ..., n).$$

Since the $\tilde{x}_i^*$ were arbitrary, M' is interpolating for y'. But since $y' \notin M'$, this is impossible by the first part of the proof, and hence the proof is complete.

Finally, we cite a result of Ault [2] which is in direct contrast to Theorem 3.6. Let Σ denote any subset of the set $\{z \in \mathcal{C}: |z| = 1\}$. We say that an n-dimensional subspace M of l_1^m is *interpolating relative to* Σ if for each $x \in l_1^m$ and each set of n linearly independent functionals $x_1^*, ..., x_n^*$ in ext $S((l_1^n)^*)$—whose coordinates lie in Σ—there exists $y \in M$ such that $x_i^*(y) = x_i^*(x)$ $(i = 1, ..., n)$. Note that a finite dimensional subspace of l_1^m is interpolating relative to $\Sigma = \{z: |z| = 1\}$ if and only if it is an interpolating subspace.

Using a Baire category argument, Ault proved the following.

THEOREM (see [2, Corollary 1.21]). *Let Σ be any countable subset of* $\{z: |z| = 1\}$. *Then for each $1 \leqslant n \leqslant m$, the complex space l_1^m contains an n-dimensional subspace which is interpolating relative to Σ.*

REFERENCES

1. D. A. AULT, F. R. DEUTSCH, P. D. MORRIS, AND J. E. OLSON, Interpolating subspaces in approximation theory, *J. Approximation Theory*, **3** (1970), 164–182.
2. D. A. AULT, Characterization theorems and algorithms for best approximations, Ph.D. thesis, The Pennsylvania State University, 1970.